Digital Signal Processing: From Complex Numbers to the Hilbert Transform

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***Abstract*—We present a bottom up review of complex numbers through analytical functions commonly used in digital signal processing. Standard practice uses complex exponential functions for engineering and physics problems including differential equations, electrical engineering, analog and digital signal processing, control systems, mechanical vibrations, and wave propagation. We often use these methods by rote. In this paper, we review complex math and aim to fill common knowledge gaps in digital signal processing.**

Keywords— Digital signal processing, complex exponential, analytic function, Hilbert transform, software defined radio

I. Introduction

Complex numbers, exponential functions, and analytical functions are the fundamental mathematics for digital signal processing (DSP). We often apply complex numbers by rote without giving the special properties of complex numbers much thought. Complex exponential functions have several names thanks to their many useful properties.

Following the fundamentals from baseball, if you can't make the simple catch, you won't last long in big league baseball. We will review the fundamentals of complex numbers, and hopefully, improve understanding. With a strong foundation, the properties of complex exponentials can be effectively leveraged. In this paper, we have included a number of refinements and additions over the technical report appendix in [1], covering operator overloading, direction vectors, insight into the square root definition, and additional material on complex exponential functions. "… in mathematics, you don't understand things. You just get used to them." John von Neumann [2]. Now, that we are used to solving problems using complex exponentials; hopefully, we can re-tell the story and improve comprehension.

II. Number Line and Distance

Following the baseball fundamental analogy, we start from the fundamentals and return to the simple number line. We will take a detour and use East and West number line to provide a foundation for the algebra class engraved fact: $(-1)(-1) = +1$. The East and West direction vectors provide a basis for the $+$ and $-$ direction vectors used on a one dimensional number line. The number line examples help bridge the gap to introduce the imaginary number, $j = \sqrt{-1}$, in section III.

A. Distance and the Number Line

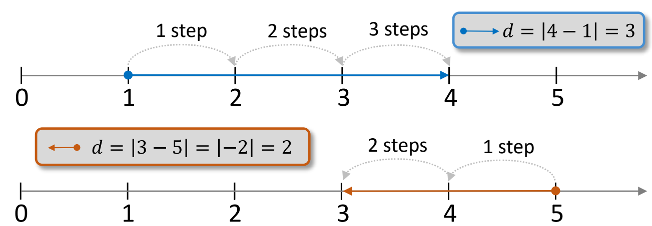
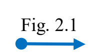


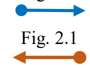
Fig. 2.1. Number Line and Distance.

Our common sense idea of distance is the number of steps from point A to point B. In Fig. 2.1, we provide two examples of distance, which is always a positive number. Keep in mind that rulers measure distance in cm or inch scales without regard to negative numbers. For example, (2.1) defines the distance between points A and B on the number line. In (2.2) stepping a little ahead, we have $3 - 5 = -2$. Comparing the arrows in Fig. 2.1, we see that for $d = |3 - 5| = |-2| = 2$, we have 2 steps in the opposite direction (to the left). Fig. 2.1 introduces direction vectors for positive and negative numbers. We will show the imaginary number, $j = \sqrt{-1}$, is a direction vector like $(+1)$ and (-1).

$$d = |B - A| = |4 - 1| = 3 \tag{2.1}$$



$$d = |3 - 5| = |-2| = 2 \tag{2.2}$$



The math operators, $+$ and $-$, are overloaded. Here, we have borrowed a concept from computer science. An overloaded operator (or function) has multiple definitions depending on its context. We use $+$ and $-$ symbols for addition and subtraction. We also use $+$ and $-$ symbols to indicate direction (for $+$, the direction is to the right, and for $-$, the direction is to the left). In order to help separate the overloaded operations, we will start with a number line using East and West directions. This extra step will help explain the entrenched algebra equation: $(-1)(-1) = +1$. The East-West number line will also help explain, the definition for square root and $j = \sqrt{-1}$ imaginary number definition.

B. Direction Vectors

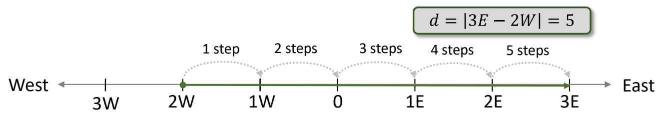


Fig. 2.2. East-West Number Line Example.

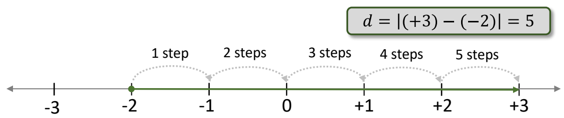


Fig. 2.3. Rewritten Fig. 2.2 using + for *East* and – for *West*.

Fig. 2.2 shows a number line with East and West directions: East to the right and West to the left. Although this is simple math, we intend to use a similar explanation to define the imaginary number $j = \sqrt{-1}$. Operators $+$ and $-$ are overloaded since, we use the $+$ symbol for addition $(5 + 3)$ and direction to the right ($+4$ on a number line), and the $-$ symbol for subtraction $(7 - 2)$ and direction to the left (-9 on a number line). Equation (2.3) shows the distance from point $2W$ (2 West) to point $3E$ (3 East). From the number line, the distance is 5.

$$d = |3E - 2W| = 5 \quad \text{Fig. 2.2} \tag{2.3}$$

Since West is the opposite direction of East, we can define West as $West = -East$ (see section II.A, and Fig. 2.1-2.3). In Equation (2.4), we replace W with $-E$. The length of the direction vector E is $|E| = 1$. In (2.4), we have $-2(-E)$ where -2 is subtraction and $(-E)$ is the opposite direction of East. We can rearrange $-2(-E)$ to $2(-1)(-1)E = 2(-1)^2E$. We know the distance is 5, so $(-1)^2E$ is equivalent to addition, $+E$, in (2.4). Fig. 2.3 and (2.5) help separate addition and subtraction from the direction vectors $(+1)$ and (-1). Building on the concept of operator overloading and direction vectors, we will introduce the imaginary number and show $j = \sqrt{-1}$ is a direction vector like $(+1)$ and (-1).

$$\begin{aligned} d &= |3E - 2W| = |3E + [-2(-E)]| \\ &= |3E + (-1)(-1)2E| = |3E + 2(-1)^2E| \\ &= |3E + (+1)2E| = |5E| = 5|E| = 5 \cdot 1 = 5 \end{aligned} \quad \text{Fig. 2.2} \tag{2.4}$$

$$\begin{aligned} d &= |(+3) - (-2)| = |(+3) + (-1)^2(2)| \\ &= |(+3) + (+1)2| = |3 + 2| = 5 \end{aligned} \quad \text{Fig. 2.3} \tag{2.5}$$

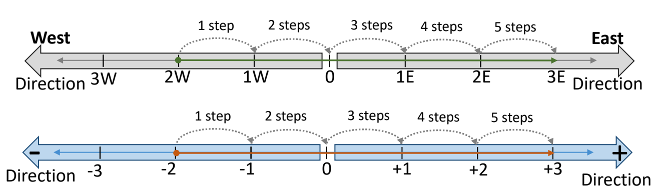


Fig. 2.4. Direction Vectors' Summary

We summarize number line direction vectors in Fig. 2.4. We can locate points using East or West of the origin (0 position) or $+$ or $-$ directions from the origin. Fig. 2.4 shows the distance is 5 from $2W$ to $3E$ and the distance is 5 from -2 to +3. Equations (2.4) and (2.5) emphasize direction vectors (East and West) and $(+1$ and $-1)$, and summarize computing distance.

C. Complex Number Introduction

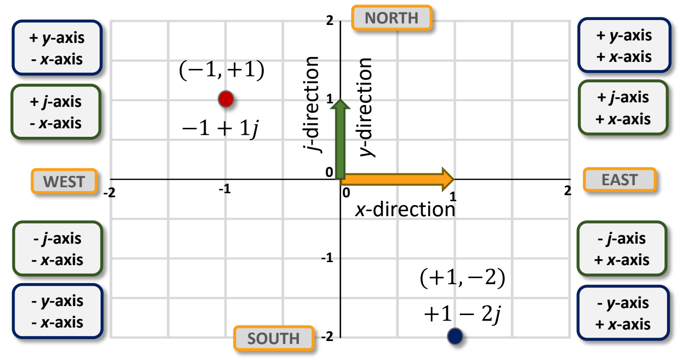


Fig. 2.5. Coordinate System Comparison

In Fig. 2.5, we extend the 1-dimensional number line to a 2-dimensional (Cartesian) plane. We show three different ways to represent points using map directions (North, South, East, and West), (x-axis, and y-axis) and complex number format. For the points located using x-axis and y-axis, the directions +/– refer to right/left or up/down. The x-axis direction vector is represented by $\vec{x}$ and the y-axis is $\vec{y}$. The x-axis and y-axis direction vectors both have unit length. We use notation 1_x and 1_y to emphasize that the x-axis and y-axis direction vectors have a length of 1.

$$(x, y) = (-1, +1) = -1\vec{x} + 1\vec{y} = (1\,West, 1\,North) \tag{2.6}$$

$$x + j\,y = -1 + 1j \tag{2.7}$$

$$(x, y) = (+1, -2) = +1\vec{x} - 2\vec{y} = (1\,East, 2\,South) \tag{2.8}$$

$$x + j\,y = +1 - 2j \tag{2.9}$$

Fig. 2.5 locates points in (2.6)-(2.9) using all three coordinate systems. For example, the point, ● $(x, y) = (-1, +1)$, is mapped to $-1\vec{x} + 1\vec{y}$ and $(1\,West, 1\,North)$. The complex number, $-1 + 1j$, also maps to the same point. The imaginary number $j = \sqrt{-1}$ is a direction vector like North, South, East, West, $(+1)$, (-1), $\vec{x}$, or $\vec{y}$. The length or magnitude of $j = \sqrt{-1}$ is also 1. In section III, we will show the imaginary number, $j = \sqrt{-1}$, is a direction vector with a set of useful properties.

In summary, $j = \sqrt{-1}$, $(-j)$, $(+1)$ and (-1) are direction vectors. We have already looked at the two properties: $(+1)(+1) = (+1)^2 = +1$ and $(-1)(-1) = (-1)^2 = +1$. In section III.C, we will consider the properties of the imaginary direction vector: $j^1, j^2, j^3, j^4 \cdots$.

III. Complex Numbers

The complex number, $u = +4 + 3j$, contains two parts. The number +4 is a real number just like the real numbers on the East-West, or +/– number lines in Fig. 2.4. The imaginary number $+3j$ also contains the real number $+3$. In this section, we will define the square root of (-1) as $j = \sqrt{-1}$ using the direction vectors covered in section II. We first look at the definition of the square root in Fig. 3.1 and how the square root function, $y(x) = \sqrt{x}$, in (3.1) maps positive x-values to positive y-values. The mapping is the key to understanding imaginary numbers. We will show complex numbers are 'mostly harmless' [3] and very useful for DSP.

A. Square Root

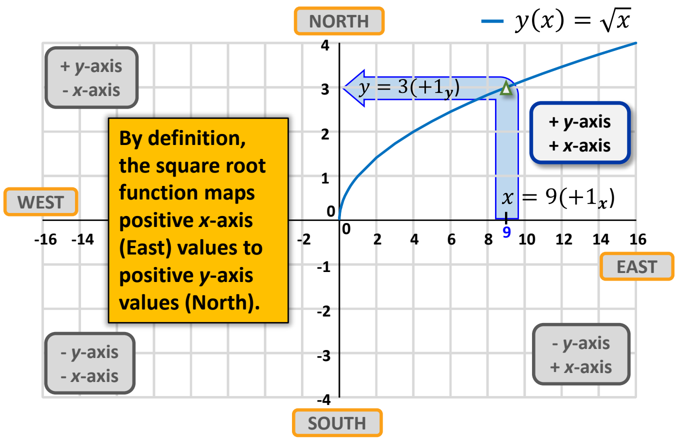


Fig. 3.1. Graph for $y(x) = \sqrt{x}$ for $x \geq 0$

By definition, the square root function in Fig. 3.1 and (3.1) maps positive x-axis (East) values to positive y-axis values (North). The x-axis (1_x) and y-axis (1_y) direction vectors will help explain the definition for the square root function, $\sqrt{x}$.

We write the square root equation, $y(x) = \sqrt{x}$, in a more general form using direction vectors in equation (3.2). Equation (3.2) will be used to define the square root of a negative number. For $x = +9$, we write $x = 9(+1_x)$ where $+1_x$ represents the positive x-axis direction vector, the square root function in equation (3.3) gives $y(x) = 3(+1_y)$ where $(+1_y)$ is the positive y-axis direction vector. The square root function maps $(+1_x)$ to $(+1_y) = \sqrt{(+1_x)^2}$ as illustrated in Fig. 3.1.

$$y(x) = \sqrt{x} \text{ for } 0 \leq x < \infty, \text{ gives } \; 0 \leq y < \infty \tag{3.1}$$

$$y(x) = \sqrt{x(+1_x)} \qquad \text{(more general definition)} \tag{3.2}$$

Mapping from Fig. 3.1 showing (+x axis) mapped to (+y axis)

$$y(x) = \sqrt{x(+1_x)} = \sqrt{9(1_x)(1_x)} = \sqrt{9}\sqrt{(1_x)^2} = 3(1_y) \tag{3.3}$$

$x = +9$ — More General Definition — $+9$ — Definition of $\sqrt{\ }$: $(+1_y) \equiv \sqrt{(+1_x)^2}$ — y-axis Direction vector

B. Imaginary Direction Vector

In Fig. 3.1, we see that the square root function maps positive x-axis values to the positive y-axis values. Following the intuition in Fig. 3.1, we want the square root function to map negative x-axis values to a *new* positive axis (see Fig. 3.2).

We reconsider the square root function in (3.4). Remember, positive x-axis values, $(+1_x)$, are mapped to positive y-axis values $1_y = \sqrt{(+1_x)^2}$. Negative x-axis values, (-1_x), are mapped to positive values on a new axis (z-axis): where $z = \sqrt{(-1)(1_x)^2}$ and $\sqrt{(-1)(1_x)^2}$ is the new direction vector. We define the new direction vector as $j \equiv \sqrt{(-1)(1_x)^2}$. In Equation (3.5), we rewrite the square root Equations (3.3) and (3.4) in standard form. The constant, $j = +\sqrt{-1}$, is a direction vector with useful properties.

Mapping from Fig. 3.2 showing (+x axis) mapped to (+j axis)

$$y(x) = \sqrt{4(-1_x)(+1_x)} = \sqrt{4[(-1)(1_x)^2]} = 2(j) \tag{3.4}$$

$x = -4$ — Definition of $\sqrt{-1}$: $(+j) \equiv \sqrt{-1(1_x)^2}$

$$y(x) = \begin{cases} \sqrt{x} & \text{for } 0 \leq x < \infty \\ j\sqrt{|x|}, & \text{for } -\infty < x < 0 \end{cases} \tag{3.5}$$

where $j = \sqrt{-1}$ is a direction vector

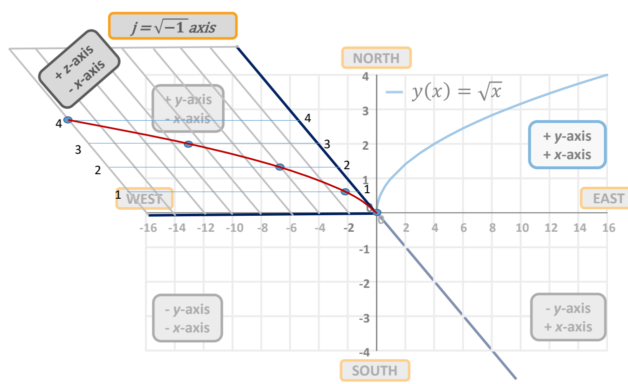


Fig. 3.2. Graph for $y(x) = \sqrt{x}$ for $x < 0$.

In Fig. 3.2, we add a z-axis for the imaginary axis $j = +\sqrt{-1}$. We see the square root function maps negative x values ($-\infty < x < 0$) to the positive imaginary axis $j = +\sqrt{-1}$ (z-axis). For negative x values ($-\infty < x < 0$), we have $y(x) = j\sqrt{|x|}$. For positive x values, $0 \leq x < \infty$, we have the standard square root function, $y(x) = \sqrt{x}$. For positive x values, we have x-axis $(+1_x)$ direction vector mapped to y-axis $(+1_y)$ direction vector. For negative x values, $-\infty < x < 0$, we have x-axis (-1_x) direction vector mapped to z-axis $(+j)$ direction vector. Fig. 3.2 shows direction vectors simplify understanding imaginary numbers. In summary, square root function preserves the mapping of x-axis values to positive y-axis and to positive imaginary axis values: $+1_x$ is mapped to $+1_y$ and -1_x is mapped to $+j$.

The complex number in (3.6) consists of two parts: a real number, -4, + (addition) and an imaginary number, $+3j$. The imaginary number $+3j$ also contains the real number $+3$. A similar coordinate is the (x, y) Cartesian coordinate $(-4, +3)$. In Fig. 2.5, we compared Cartesian coordinates to complex numbers. Cartesian coordinates have $\vec{x}$ and $\vec{y}$ direction vectors. Complex numbers have $\vec{x}$ direction and j direction vectors.

$$u = -4 + (+3j) = -4 + 3j \quad \text{Complex number consisting of a real number } (-4) \text{ and imaginary number } (+3j). \tag{3.6}$$

In the 1600's, the constant, $j = +\sqrt{-1}$, was unfortunately named imaginary number. 'Imaginary' numbers are very useful for math and engineering. We will show the complex exponential, $e^{j(\omega t+\varphi)}$, has several useful properties for digital signal processing.

C. Imaginary Number Properties

We begin by looking at the periodic properties of $(+1)^n$ and $(-1)^n$. In (3.7), we see that $(+1)^n$ is a constant. In (3.8), we show $(-1)^n$ is periodic with $(-1)^n = +1, -1, +1, -1 \cdots$ with a cycle length of 2. On a number line there are two direction vectors, (+1), and (-1).

The function j^n in Equation (3.9) is periodic; $j^n = +1,\ +j,\ -1,\ -j,\ +1, \cdots$ with a cycle length of 4. For a complex number, we need 4 direction vectors, (+1), (-1), (+j), and (-j) to plot a point on the plane. We can compare the direction vectors for a number line to the direction vectors for the complex plane. The function $(-1)^n$ has a cycle length of 2 and describes the two direction vectors, (+1) and (-1) in (3.8). The function j^n has a cycle length of 4 and describes the four direction vectors for the complex plane in (3.9).

$$(+1)(+1)\cdots(+1) = (+1)^n = +1 \tag{3.7}$$

$$(-1)(-1)\cdots(-1) = (-1)^n = \begin{cases} +1 & n = \text{even} \\ -1 & n = \text{odd} \end{cases} \tag{3.8}$$

2 Direction Vectors for number line

4 Direction Vectors for complex plane

$$\begin{aligned} j^0 &= +1 \\ j^1 &= +j \\ j^2 &= \left(\sqrt{-1}\right)^2 = -1 \\ j^3 &= i^2 i = (-1)j = -j \\ j^4 &= j^2 j^2 = (-1)(-1) = +1 \\ j^5 &= j^1 j^4 = j(+1) = +j \\ &\vdots \end{aligned} \qquad j^n = \begin{cases} +j & n = 1, 5, 9, \cdots \\ -1 & n = 2, 6, 10, \cdots \\ -j & n = 3, 7, 11, \cdots \\ +1 & n = 0, 4, 8, \cdots \end{cases} \tag{3.9}$$

IV. Complex Exponential

Complex exponentials have two very useful properties for digital signal processing. First, frequency translation and phase shift only require a multiplication. Second, integration and derivative operations are simple for complex exponential functions. The importance of the complex exponential is found in the equation, $e^{j\pi} + 1 = 0$. It summarizes five of the most important concepts in mathematics: 0, 1, $j = \sqrt{-1}$, π, and e^x function. The imaginary number can also be written in terms of a complex exponential, $j = e^{j\frac{\pi}{2}}$. A standard time domain function, $x(t)$, is not a complex exponential function. The Hilbert transform converts a standard time domain function, $x(t)$, into an analytic function. An analytic function has the same frequency translation and phase shift properties as a complex exponential function.

We present an introduction to complex exponential functions in section IV.A followed by properties of complex exponentials in section IV.B. Section IV.C covers rotating vectors for steady state alternating current circuits (forced differential equations). Section IV.D introduces analytical functions. Analytical functions have the same frequency translation and phase shift properties as complex exponential functions. Section IV.D introduces the Hilbert transform which converts a real time domain function into an analytic function. We present a simple software defined radio example using the Hilbert transform and frequency translation in section V.

A. Complex Exponential Introduction

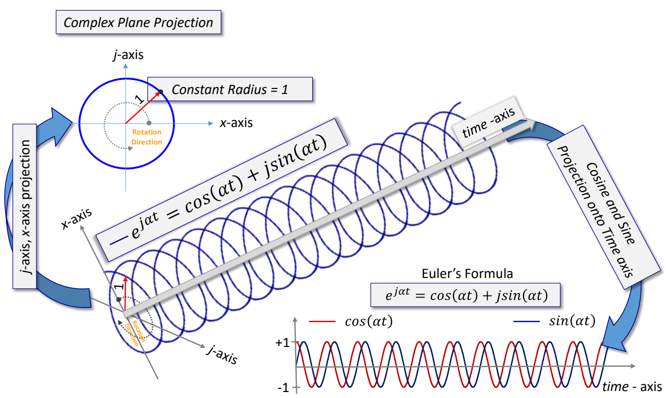


Fig. 4.1. Complex Exponential Function.

Fig. 4.1 illustrates several properties of the complex exponential in (4.1)-(4.3). The center of the figure shows the complex exponential maps out a helix in 3 dimensions. By projecting the helix onto the complex plane, we see $e^{j\alpha t}$ has a radius or magnitude of 1. For the time axis projection, we see the cosine and sine terms from Euler's formula in (4.3). In Equation (4.2), the phase term is replaced by the complex constant A. We will revisit (4.2) in section IV.B on rotating vectors.

$$x(t) = e^{j(2\pi f t + \varphi)} \tag{4.1}$$

Frequency Phase

$$x(t) = e^{j2\pi f t} e^{j\varphi} = A e^{j2\pi f t} \tag{4.2}$$

Complex constant contains Phase term

Frequency Phase

$$x(t) = e^{j(2\pi f t+\varphi)} = \underbrace{\cos(2\pi f t + \varphi) + j\sin(2\pi f t + \varphi)}_{\text{Euler's Formula}} \tag{4.3}$$

We can derive Euler's formula in (4.3) using the infinite series for sine in (4.4), cosine in (4.5) and e^x in (4.6). In (4.7), we complete the derivation of Euler's formula using the properties of j^n from section III.C.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{4.4}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{4.5}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{4.6}$$

$$e^{j\alpha t} = 1 + j\alpha t + \frac{(j\alpha t)^2}{2!} + \frac{(j\alpha t)^3}{3!} + \frac{(j\alpha t)^4}{4!} + \frac{(j\alpha t)^5}{5!} + \cdots \tag{4.7}$$

$$e^{j\alpha t} = \left[1 + \frac{(j\alpha t)^2}{2!} + \cdots\right] + \left[j\alpha t + \frac{(j\alpha t)^3}{3!} + \cdots\right]$$

$$e^{j\alpha t} = \left[1 - \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^4}{4!} - \cdots\right] + j\left[\alpha t + \frac{j^2(\alpha t)^3}{3!} + \cdots\right]$$

$$e^{j\alpha t} \cos(\alpha t) + j\left[\alpha t - \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^5}{5!} + \cdots\right]$$

$$e^{j\alpha t} = \cos(\alpha t) + j\sin(\alpha t) \qquad \text{also see (4.3)}$$

B. Complex Exponential Properties

The complex exponential has several useful properties. Integration and differentiation are simple for the complex exponential function. For digital signal processing, frequency translation and phase shift only require a multiplication. The frequency and phase shift properties greatly simplify digital signal processing. We will present a short software defined radio example in section V using complex exponentials and the Hilbert transform.

The derivative of a complex exponential is a constant times the original function as shown in (4.8). For integration, (4.9) shows the integral is also a constant times the original function plus a constant of integration. Rotating vectors in section IV.C make use of the constant times the original function property from (4.8).

$$\text{For } x(t) = e^{j\alpha t}, \ \tfrac{d}{dt}[x(t)] = j\alpha e^{j\alpha t} = j\alpha x(t) \tag{4.8}$$

$$\text{For } x(t) = e^{j\alpha t}, \ \int x(t)\,dt = \frac{1}{j\alpha}e^{j\alpha t} + C = \frac{1}{j\alpha}x(t) + C \tag{4.9}$$

Equation (4.10) illustrates complex exponentials and multiplication. The product of two complex signal sources creates a single sum frequency term, $(f_1 + f_2)$. The product of two cosine functions in (4.11) produces a more complicated result with sum, $(f_1 + f_2)$, and difference, $(f_1 - f_2)$, frequency terms. Phase shift in (4.10) only requires a complex multiplication. If DSP data is in terms of complex exponentials, frequency translation and phase shift are straightforward, as illustrated in Fig. 4.1. The product of complex signal sources is a single frequency sum term. Phase shift only requires multiplication by a complex constant.

Complex Signal Sources; Sum of Frequency Terms

$$e^{j(2\pi f_1 t)} \cdot e^{j(2\pi f_2 t)} \cdot e^{j\varphi} = e^{j[2\pi(f_1+f_2)t+\varphi]} \tag{4.10}$$

Phase Shift; Phase Shift

$$\cos(2\pi f_1 t)\cos(2\pi f_2 t) = \tfrac{1}{2}[\cos(2\pi(f_1-f_2)t) + \cos(2\pi(f_1+f_2)t)] \tag{4.11}$$

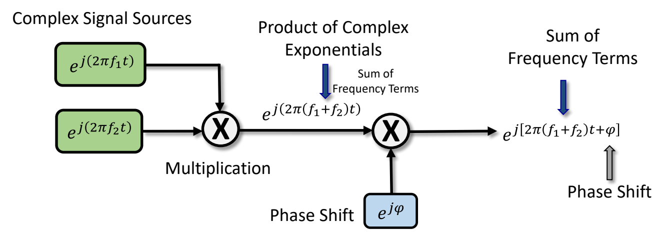


Fig. 4.1. Frequency Translation and Phase Shift

C. Rotating Vectors

A rotating vector is another name for a complex exponential. For steady state electrical circuits, rotating vectors, or phasors, reduce solving a circuit problem to algebra. We start from a differential equation in (4.12) with an input of the form $v_{in}(t) = Acos(\omega t + \phi)$ in (4.13).

$$\frac{d^2}{dt^2}v_{out}(t) + 4\frac{d}{dt}v_{out}(t) + 3v_{out}(t) = v_{in}(t) \tag{4.12}$$

$$v_{in}(t) = \sqrt{2}cos\left(2t + \frac{\pi}{4}\right) \Rightarrow V_{in}(j\omega) = \frac{A}{\sqrt{2}}e^{j(\omega t+\phi)} \Rightarrow$$
$$V_{in}(j\omega) = e^{j\frac{\pi}{4}} \quad \text{(convert } v_{in}(t) \text{ to phasor form)} \tag{4.13}$$

We convert $v_{in}(t)$ into the phasor, $V_{in}(j\omega) = e^{j\frac{\pi}{4}}$, using the complex exponential-cosine transform property in (4.13). Since the radian frequency is a constant, the $e^{j\omega t}$ term does not affect the amplitude or phase. We drop the $e^{j\omega t}$ term to save ink.

Using the property for complex exponentials in (4.8), we convert (4.12) into phasor form in (4.14). We solve for V_{out} in (4.15) where $\omega = 2\frac{radians}{sec}$. In (4.16), we convert the $V_{out}(j\omega)$ phasor form back into the time domain, $v_{out}(t)$.

$$(j\omega)^2V_{out}(j\omega) + 4(j\omega)V_{out}(j\omega) + 3V_{out}(j\omega) = V_{in}(j\omega)$$
$$\text{(phasor equation form)} \tag{4.14}$$

$$V_{out}(j\omega) = \left.\frac{V_{in}(j\omega)}{(3+4j\omega-\omega^2)}\right|_{\omega=2} = \frac{e^{j\frac{\pi}{4}}}{(-1+8j)} = 0.124e^{+j2.23} \tag{4.15}$$

$$V_{out}(j\omega) = 0.124e^{j0.710\pi} \Rightarrow v_{out}(t) = 0.124\sqrt{2}cos(2t + .710\pi)$$
$$\text{(convert } V_{out} \text{ to time domain)} \tag{4.16}$$

We have presented a brief introduction to rotating vectors or phasors in section IV.C. Understanding the properties of complex exponential functions necessarily supports the understanding of rotating vectors.

D. Hilbert Transform and Analytic Functions

In Fig. 4.2, we convert a simple cosine function into an analytic function. An analytic function only contains positive complex exponential terms, $e^{+j\alpha t}$. Real functions, like cosine, have symmetric complex exponential terms, $e^{+j\omega t}$ and $e^{-j\omega t}$ (Fourier transform is symmetric). To convert a general real function, $r(t)$, into an analytic function, we "simply" zero out the negative complex exponential terms in $R(j\omega)$ as illustrated in Fig. 4.3. Fig. 4.4 shows a Hilbert filter implemented as a low pass digital filter to convert a real function, $r(kT)$, into an analytic function, $a(kT)$.

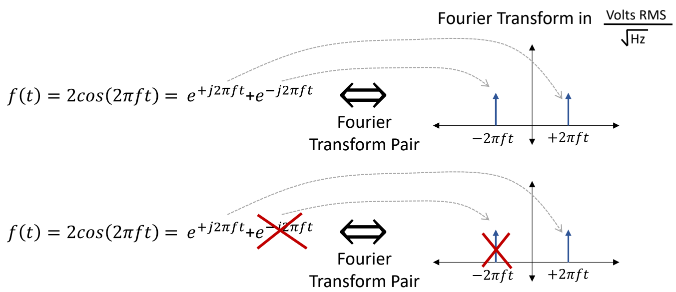


Figure 4.2. Analytic Signal Transform

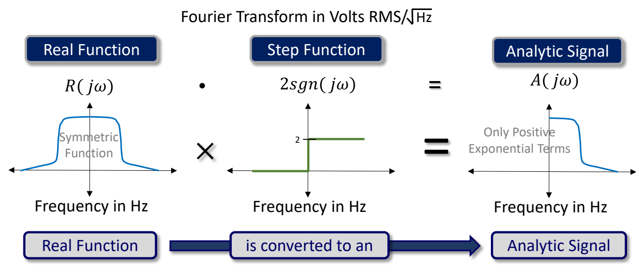


Figure 4.3. Real to Analytic Signal Transform

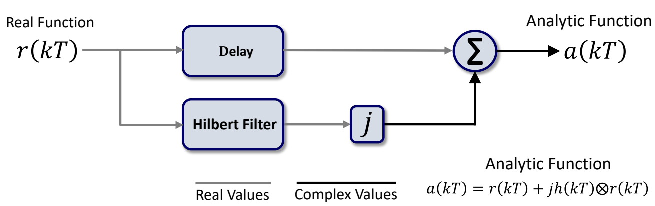


Fig. 4.4. Real Function to Analytic Function Block Diagram

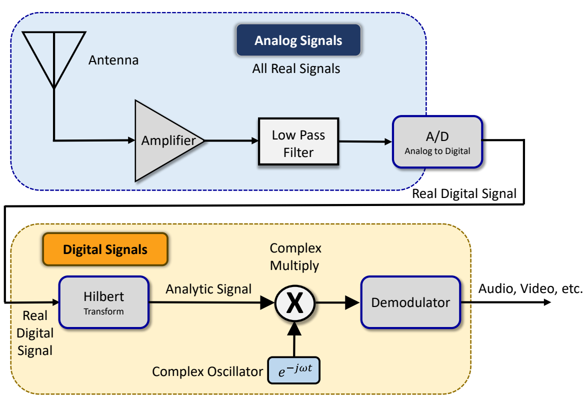


Fig. 5.1. Simplified Software Defined Radio Architecture

V. Software Defined Radio

Important features for a basic software defined radio are presented in Fig. 5.1. A Hilbert transform is used to convert the output of the analog-to-digital converter to an analytic signal. A complex multiply is used for frequency translation. Audio, video, internet packets, etc. are recovered from the demodulation stage.

VI. Conclusion.

We have reviewed the fundamental mathematics for digital signal processing (DSP): complex numbers, exponential functions and analytical functions. We presented a review of direction vectors to provide a solid foundation to explain imaginary numbers. Using the direction vectors, (+1), and (-1), we show $j = \sqrt{-1}$ is another direction vector. We also show that the direction vector $j = \sqrt{-1}$ extends the connection between (-1) and $(+1)$ direction vectors. The cycle length for $(-1)^n$ sequence is 2. We show the cycle length of $(j)^n$ is 4 and how it is an extension of $(-1)^n$.

With a solid foundation, we only need a few paragraphs to cover the properties of complex exponentials, and review rotating vectors. We have illustrated the utility of complex exponential functions for digital signal processing in section IV. The paper concludes with a brief introduction to the Hilbert transform in Fig. 4.4 and some software defined radio fundamentals in Fig. 5.1.

VII. Disclaimer and Copyright

VIII. References

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